A Plethora of Strange Nonchaotic Attractors

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(February 3, 2008)

Abstract

We show that it is possible to devise a large class of skew–product dynamical systems which have strange nonchaotic attractors (SNAs): the dynamics is asymptotically on fractal attractors and the largest Lyapunov exponent is nonpositive. Furthermore, we show that quasiperiodic forcing, which has been a hallmark of essentially all hitherto known examples of such dynamics is *not* necessary for the creation of SNAs.

I. INTRODUCTION

Since 1984, when Grebogi, Ott, Pelikan, and Yorke [1] described attractors that were strange but not chaotic, interest in these exotic objects has been increasing [2]. Several studies, both theoretical [3–9] and experimental [10–15], have, over the years, elucidated the principal features of such attractors which appear to be *generic* in quasiperiodically driven nonlinear dynamical systems. These are geometrically strange sets (fractals) on which *all* Lyapunov exponents are either zero or negative. Further, owing to their fractal character, motion on SNAs is intermittent and aperiodic.

A very important connection between classically driven systems and the Schrödinger equation for a particle in a quasiperiodic potential [16] makes the correspondence between nonchaotic attractors and localized states [17–19], and this further underscores the interest in strange nonchaotic attractors (SNAs). These results show that the transition from an invariant circle to SNA in the iterative mapping is related to the transition from extended to localized states in the quantum system.

Although there are many known examples of systems with SNAs, proving the existence of such attractors is a mathematically nontrivial task. Indeed, rigorous results exist for only two systems—the original system introduced by Grebogi et al., and the Harper map [17]. Keller [20] and Bezhaeva and Oseledets [21] have shown in the dynamical system [1]

$$x_{i+1} = 2\alpha \cos 2\pi\phi_i \tanh x_i \tag{1}$$

$$\phi_{i+1} = \{\omega + \phi_i\}, \tag{2}$$

(we use the notation $\{y\} \equiv y \bmod 1$) that for α sufficiently large and for ω an irrational number, the attractor is fractal, has a singular–continuous spectrum, and is the support of an ergodic SRB measure [22]. For the Harper map (E being a parameter)

$$x_{i+1} = -[x_i - E + 2\alpha \cos 2\pi\phi_i]^{-1} \tag{3}$$

$$\phi_{i+1} = \{\omega + \phi_i\}, \tag{4}$$

a persuasive argument [17] suggests that, again for α sufficiently large and ω irrational, the attractor of the dynamics is fractal, and the dynamics is nonchaotic since the Lyapunov exponent is nonpositive [18].

Virtually all known examples of systems with SNAs—and the two cases above are typical—have the skew–product [1] form and have a quasiperiodic driving term. How necessary are these features? This question has considerable practical relevance since there are known cases of *experimental* systems [14,15] where the dynamics appears to be on strange nonchaotic attractors, but where there is no quasiperiodic driving. Furthermore, it is also known that a judicious choice of forcing term can convert chaotic dynamics to nonchaotic dynamics [23–26]. Are the resulting attractors SNAs? In particular, can SNAs be formed via non-periodic (and also not quasiperiodic) driving of a nonlinear system?

In the present paper we address two issues that are of concern in the study of SNAs. The first deals with the ubiquity of such attractors: what are the necessary conditions for a dynamical system such that SNAs result? While we do not provide an exhaustive answer, our study shows that strange nonchaotic dynamics can be realized in a wide variety of systems, and may in fact be quite common. Our examples all remain, however, within the skew–product paradigm. The second issue relates, as discussed above, to the necessity of quasiperiodic forcing and we find that it is possible to construct dynamical systems wherein there is *no* explicit quasiperiodic forcing and the dynamics is on SNAs.

These results are described in the following two sections of this paper. In Section 2, we extend the general arguments which establish the existence of SNAs in Eqs. (1) and Eq. (3) so as to yield a large variety of dynamical systems wherein the motion will remain on SNAs. In Section 3, we further generalize the dynamics so that the forcing is no longer quasiperiodic but the resulting dynamics continues to be on SNAs. This is followed by a brief summary

[1] In a skew–product system, one of the variables evolves entirely independently of the other(s), as ϕ does in these examples.

and discussion in Section 4.

II. STRANGE NONCHAOTIC DYNAMICS

By now there is a host of examples of strange nonchaotic dynamics [2] in driven systems. These include both flows such as the driven Duffing oscillator [27,28] and iterative maps such as the driven logistic map [29,30]. Most studies to date have relied on numerical techniques to establish the presence of SNAs, by explicitly computing the Lyapunov exponents and by determining the fractal dimension. SNAs are "intermediate" between strange chaotic attractors and nonchaotic regular dynamics with regard to many dynamical and structural properties, some of which can be determined through study of power–spectra, correlation functions, and related measures (see Ref. [2] for a recent review). The most extensively studied cases all have a skew–product dynamical structure, and thus these are systems in $n+1$ dimensions, the latter dimension pertaining to the dynamics of the driving term.

In the present paper, we study the case of driven iterative maps in 1+1 dimension for convenience. Rigorous mathematical results concerning SNAs are so far available only for the simplest such examples. However, the arguments presented here largely carry over to the case of flows as well, and can be extended to higher dimensions, so this is not a restriction.

A. Generalizing the original SNA system

The following heuristic arguments, first put forward by Grebogi et al. [1] for the system embodied in Eq. (1) suggest that SNAs should occur for appropriate values of α.

The mapping $x \to 2\alpha \tanh x$ is 1–1 and contracting, taking the real line into the interval $[-2\alpha, 2\alpha]$. Because ω is an irrational number, the dynamics in ϕ [cf. Eq. (2)] is ergodic in the unit interval. The attractor of the dynamical system Eq. (1) therefore must be contained in the strip $[-2\alpha, 2\alpha] \otimes [0, 1]$. A point x_n with the corresponding $\phi_n = 1/4$ will map to $(x_{n+1} = 0, \phi_{n+1} = \omega + 1/4)$, after which subsequent iterates will all remain on the line $(x = 0, \phi)$, as will points with $\phi_n = 3/4$. This line therefore forms an invariant subspace,

but for large enough α, this subspace is transversally unstable. Thus, it follows that the attractor has a dense set of points on the line $x = 0, \theta \in [0, 1]$ (since ω is irrational), but the entire line itself cannot be the attractor for $\alpha > 1$, since the dynamics is unstable on that line. There will, therefore at some α, be a "blowout bifurcation" [8,9] transition to strange nonchaotic dynamics.

Generalizing Eq. (1) (keeping the skew–product structure intact) as

$$x_{i+1} = \alpha f(x_i) g(\phi_i) \tag{5}$$

it is clear that the same arguments will carry over so long as the following properties hold:

(i) $f(x)$ is 1-1 and contracting, with $f(0) = 0$.

(ii) $f'(0) \neq 0$, and concurrently,

(iii) $g(\phi) = 0$ for some $\phi = \phi_*$.

Then, clearly, all points x, ϕ_* will map to $0, \{\omega + \phi_*\}$, and the subsequent dynamics will be dense on the line $x = 0, \phi$. From condition (ii) above, this can be made unstable locally for sufficiently large α, and from (i), since the map is contracting, the slope $|f'| \leq 1$ almost everywhere, the Lyapunov exponent can be made negative so as to give a SNA.

B. A piecewise linear SNA

A piecewise linear strange nonchaotic attractor can be simply obtained by taking, in Eq. (5),

$$\begin{aligned} f(x) &= \alpha x \qquad && |x| \leq 1/\alpha \\ &= \mathrm{sign}(x) \qquad && |x| > 1/\alpha, \end{aligned} \tag{6}$$

$$g(\phi) = \phi - 1/2. \tag{7}$$

It is easy to verify that there is indeed a blowout bifurcation near $\alpha \approx 2.33$ (Fig. 1a), above which the attractor is strange and nonchaotic: see Fig. 1b.

C. Generalizing the Harper map

In the Harper map, on the other hand, the argument for the existence of strange nonchaotic dynamics proceeds as follows [17]. In the strong–coupling limit, $\alpha \to \infty$, Eq. (3) reduces to

$$x_{i+1} = -[2\alpha \cos 2\pi\phi_i]^{-1}. \tag{8}$$

Clearly, for ϕ_i =1/4 or ϕ_i =3/4 this gives a singularity in the neighbourhood of which the mapping locally looks like a hyperbola. Since the ϕ dynamics Eq. (2) is ergodic in the interval [0,1], the image of this singularity is dense: on every ϕ–fiber there will be a singularity, and thus the dynamics is on a strange set. By continuity, even away from the strong coupling limit but for α large enough, this argument suggests that the dynamics can be strange. (In fact, for Eq. (3), there are SNAs even at $\alpha = 1$.) For a suitable choice of function $f(x)$, the Lyapunov exponent may turn out to be negative; in such a case, the attractor is strange *and* nonchaotic. For the Harper map [18], when $E = 0$ this is indeed the situation.

Proceeding as above, one can generalize the Harper map (with the ϕ–dynamics unchanged) as

$$x_{i+1} = [f(x_i) + 2\alpha g(\phi_i)]^{-1} \tag{9}$$

where f and g are now arbitrary functions, the only additional requirement being that $g(\phi)$ should have a zero in the interval [0,1]. For suitable functions $f(x)$, the Lyapunov exponent can indeed be made zero or negative, giving therefore, SNAs.

D. SNAs of the Fibonacci chain

The Harper map [17] derives from the Harper equation [31] which is the discrete Schrödinger equation for a particle in a quasiperiodic potential,

$$\psi_{n+1} + \psi_{n-1} + V(n)\psi_n = E\psi_n, \tag{10}$$

ψ_n denoting the wave-function at lattice site n, the potential being $V(n) = 2\alpha\cos 2\pi(n\omega + \phi_0)$. The identification $\psi_{n-1}/\psi_n \equiv x_n$, connects Eq. (10) and Eq. (3). This system is known to have critical (or power–law localized) states for $\alpha = 1$, when the classical system has critical SNAs [19]. It is also known that other forms of the potential $V(n)$ support critically localized states [32–35], one example being the Fibonacci chain with

$$V(n) = \alpha \qquad 0 \leq \{n\omega\} \leq \omega \tag{11}$$

$$= -\alpha \qquad \omega < \{n\omega\} \leq 1, \tag{12}$$

when all states are critical for any α.

The classical map corresponding to this potential is

$$x_{i+1} = -[x_i - E + V(i)]^{-1} \tag{13}$$

$$\phi_{i+1} = \{\omega + \phi_i\}, \tag{14}$$

where $V(i)$ is given by Eq. (11), and there is an additional overall phase ϕ_0. It is a simple task to compute the Lyapunov exponent for this mapping (Fig. 2a); if E is an eigenvalue, then the Lyapunov exponent is zero. Comparing Eq. (9) with Eq. (13), the conditions on the functions f and g are met, and therefore, the attractors of the above system are SNAs; see Fig. 2b for an example.

III. SNAS WITHOUT QUASIPERIODIC DRIVING

It is clear from the above constructions, that there are two main ingredients in achieving SNAs in dynamical systems such as Eq. (5) or Eq. (9). Firstly, one needs some mechanism for *local* instability while maintaining global stability. The main purpose of the quasiperiodic driving, namely the ϕ dynamics which is governed by Eq. (2), is to make the instabilities dense in ϕ.

This suggests that a generalization of Eq. (5) to

$$x_{i+1} = \alpha f(x_i) g(\phi_i) \tag{15}$$

$$\phi_{i+1} = h(\phi_i), \tag{16}$$

or of Eq. (9) to

$$x_{i+1} = -[f(x_i) + \alpha g(\phi_i)]^{-1} \tag{17}$$

$$\phi_{i+1} = h(\phi_i), \tag{18}$$

where the function h is not necessarily the rigid rotation, but is otherwise such that the orbit $h^n(\phi_*)$ is dense will still yield SNAs for sufficiently large α.

To be mathematically more precise [36], we need that h be a homeomorphism with an invariant ergodic probability measure of full support, and that the h^{-1} orbit of ϕ_* be dense, in which circumstances, the resulting attractor can be shown to be a SNA, following the basic proof given by Keller [20] for the case $f \equiv \tanh$, $g \equiv \cos$ and h the irrational rigid rotation. The basic property that is required is that the mapping $h(\phi)$ take the interval $[0, 2\pi]$ into some continuous subinterval (at least), and that the orbit of a typical point should be dense in this subinterval.

It should be added that the function h should have only nonpositive Lyapunov exponents since the skew–product form for the dynamical system is being retained. There are a number of possible choices for h which are distinct from the quasiperiodic rotation, but which use related maps to generate ergodic flows. This follows from the Weyl theorem [37] which states that if $\mathcal{H}$ is a polynomial function of degree $r \geq 1$ with real coefficients $a_0, \ldots, a_r$, at least one of which is an irrational number, then the map

$$\phi_n = \{\mathcal{H}(n)\} \equiv \{a_0 + a_1 n + \ldots + a_r n^r\} \tag{19}$$

is ergodic, and the sequence $\{\phi_n\}$ is uniformly distributed in the interval [0,1]. If $\mathcal{H}$ is nonlinear, then it can be easily verified that the sequence $\{\phi_n\}$ is not quasiperiodic, but the Lyapunov exponent is zero since the map preserves distance.

An example of such a system is easy to devise. A simple choice is to take $\mathcal{H}(n) = \omega n^2, \omega$ irrational, which gives

$$x_{i+1} = \alpha f(x_i) g(\phi_i) \tag{20}$$

$$\phi_{i+1} = \{\phi_i + (2i+1)\omega\}. \tag{21}$$

The SNA which obtains for f and g given by Eqs. (6) and (7), for a suitably large value of α is shown in Fig. 3a.

Of course, it is also possible to use other mappings which generate quasiperiodic motion to determine the ϕ dynamics. Many examples of such maps are known, as for instance the diffeomorphisms

$$\phi \to \{\phi + \Omega + \epsilon a(\phi)\} \tag{22}$$

for Ω a constant, ϵ sufficiently small and a an arbitrary analytic function. The orbits are everywhere dense on the interval and typically have irrational rotation number; indeed, by Denjoy's theorem, any orientation preserving C^2 diffeomorphism of the circle is topologically equivalent to a rigid rotation [38]. In the specific case of the circle map with $a(\phi) \equiv \sin 2\pi\phi$ the parameter ranges wherein the map has irrational winding number have been comprehensively described.

Other possibilities for h exist: there are examples of integrable geodesic flows on compact manifolds which have positive topological entropy but have no positive Lyapunov exponents [36], or one can even take h to be a SNA map such as Eq. (1) itself, since that is known to provide an ergodic flow [20]. This yields, for instance, the system

$$x_{i+1} = 2\alpha\phi_i \tanh x_i \tag{23}$$

$$\phi_{i+1} = 2\beta \cos 2\pi\theta_i \tanh \phi_i \tag{24}$$

$$\theta_{i+1} = \{\omega + \theta_i\}, \tag{25}$$

which has strange nonchaotic attractors for suitable values of α and β, see Fig. 3b.

IV. DISCUSSION AND SUMMARY

Although so far all known examples of systems with strange nonchaotic dynamics have appeared to require quasiperiodic forcing, the present work shows that this is, in fact, not necessary. By suitably generalizing systems wherein SNAs are known to exist [17,18,20,21],

we have constructed dynamical systems where the motion is on strange nonchaotic attractors and there is no quasiperiodic driving. This has particular relevance to experimental observations of apparently nonchaotic attractors where the dynamics does not have explicit quasiperiodic forcing. One specific example is of a gas discharge plasma where the light flux as a function of time (with current as the driving parameter) appears to lie on a SNA. This has been deduced by attractor reconstruction and extraction of fractal dimension and Lyapunov exponents [14]. The other example [15] pertains to the effect of noise which allegedly reproduces the effect of many–frequency quasiperiodic driving.

In addition, we have also shown that strange nonchaotic dynamics can be quite common. Our approach has been heuristic: the same deconstruction also provides a prescription for obtaining a large variety of dynamical systems wherein SNAs must occur. We have shown examples of two specific systems wherein the attractors are piecewise–linear fractals. Detailed analyses of such examples may prove to be simpler than the cases studied so far.

This procedure for creating SNAs is clearly not exhaustive and there may be different generalizations that will also yield strange nonchaotic dynamics. In particular, we have chosen to stay within the "skew–product" class of mappings, although other possibilities [39] may be compatible with such dynamics as well.

Acknowledgments

We are indebted to Professor Gerhard Keller for advice and correspondence. This research is supported by the Department of Science and Technology, India.

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FIGURES

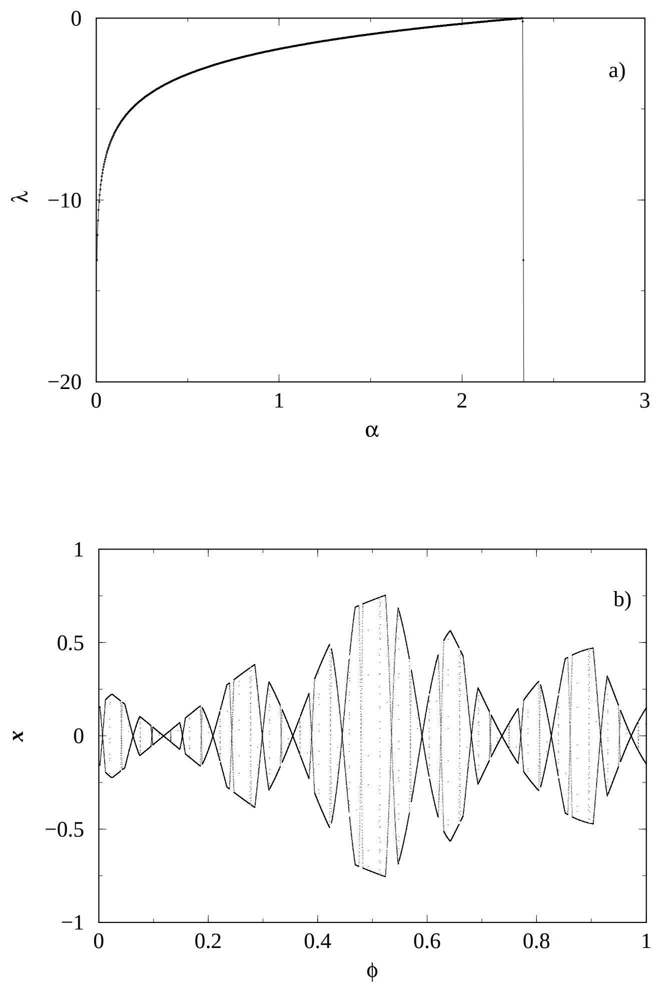

FIG. 1. The piecewise linear system with $\omega = (\sqrt{5} - 1)/2$. Plot of (a) the Lyapunov exponent as a function of the parameter α, showing the blowout bifurcation at $\alpha \approx 2.33$, above which the Lyapunov exponent $\to -\infty$, and (b) the strange non chaotic attractor for α=2.5

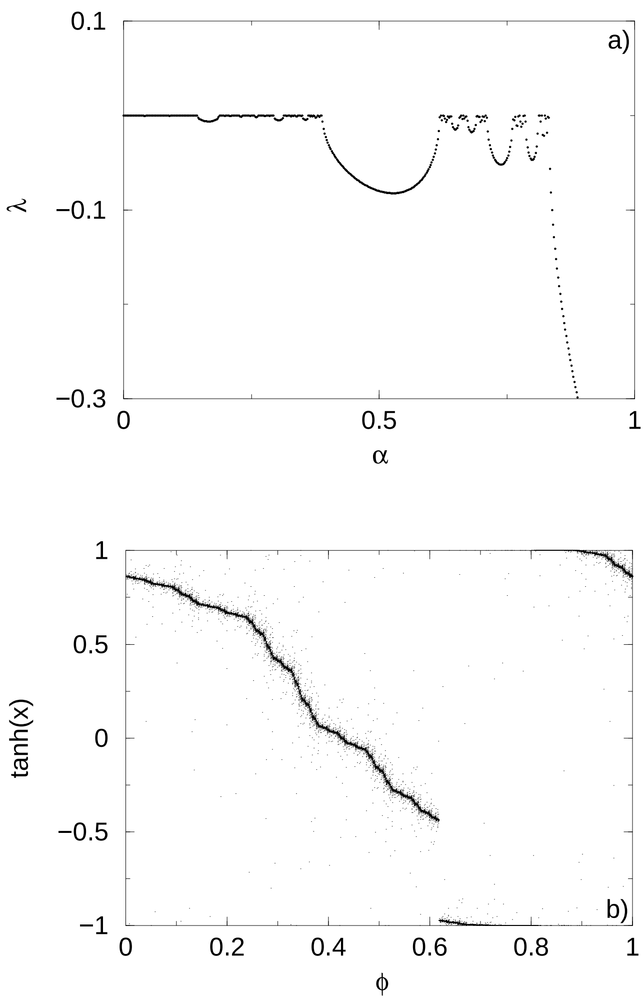

FIG. 2. The Fibonacci chain, where $\omega = (\sqrt{5}-1)/2$. (a) Variation of the Lyapunov exponent with forcing parameter α. Each zero of the Lyapunov exponent corresponds to an eigenstate in the quantum system, and a critical SNA in the classical system. (b) A typical SNA in this system, for $E = 0$, $\alpha = 0.8326745$. Note that the variable plotted on the ordinate is $\tanh x$ rather than x; this is merely for convenience.

FIG. 3. (a) SNA in the system Eq. (20) where the driving is no longer quasiperiodic. The parameters are $\alpha = 2.5, \omega$ is the golden mean ratio. (b) SNA in the system given by Eqs. (23–25), where the map governing the ϕ dynamics itself has SNA dynamics. The parameters α and β are both 2.5.(this figure submitted in gif format)